Contents

1 DFT_Gates_def_PROB Theory

Built: 15 November 2018

Parent Theories: integration_before_after, PIE_EXTREAL

1.1 Definitions

[All_distinct_events_def]

$\vdash \forall p\ L\ t.$
  All_distinct_events $p\ L\ t \iff$
  ALL_DISTINCT (MAP ($\lambda a.$ DFT_event $p\ a\ t$) L)

[ALWAYS_def]

$\vdash$ ALWAYS = ($\lambda s.$ 0)

[CSP_def]

$\vdash \forall Y\ X.$ CSP $Y\ X$ = ($\lambda s.$ **if** $Y\ s < X\ s$ **then** $X\ s$ **else** PosInf)

[D_AND_def]

$\vdash \forall X\ Y.$ D_AND $X\ Y$ = ($\lambda s.$ max ($X\ s$) ($Y\ s$))

[D_BEFORE_def]

$\vdash \forall X\ Y.$ D_BEFORE $X\ Y$ = ($\lambda s.$ **if** $X\ s < Y\ s$ **then** $X\ s$ **else** PosInf)

[D_INCLUSIVE_BEFORE_def]

$\vdash \forall X\ Y.$
  D_INCLUSIVE_BEFORE $X\ Y$ =
  ($\lambda s.$ **if** $X\ s \le Y\ s$ **then** $X\ s$ **else** PosInf)

[D_OR_def]

$\vdash \forall X\ Y.$ D_OR $X\ Y$ = ($\lambda s.$ min ($X\ s$) ($Y\ s$))

[D_SIMULT_def]

$\vdash \forall X\ Y.$ D_SIMULT $X\ Y$ = ($\lambda s.$ **if** $X\ s$ = $Y\ s$ **then** $X\ s$ **else** PosInf)

[DFT_event_def]

$\vdash \forall p\ X\ t.$ DFT_event $p\ X\ t$ = $\{s \mid X\ s \le$ Normal $t\} \cap$ p_space p

[FDEP_def]

$\vdash \forall X\ Tr.$ FDEP $X\ Tr$ = ($\lambda s.$ min ($X\ s$) ($Tr\ s$))

[HSP_def]

$\vdash \forall Y\ X.$ HSP $Y\ X$ = ($\lambda s.$ max ($Y\ s$) ($X\ s$))

[NEVER_def]

$\vdash$ NEVER = $(\lambda s.$ PosInf)

[NEVER_events_def]

$\vdash \forall X\ Y.$ NEVER_events $X\ Y \iff$ (D_AND $X\ Y$ = NEVER)

[P_AND_def]

$\vdash \forall X\ Y.$ P_AND $X\ Y$ = $(\lambda s.$ **if** $X\ s \le Y\ s$ **then** $Y\ s$ **else** PosInf)

[rv_gt0_ninfty_def]

$\vdash$ (rv_gt0_ninfty [] $\iff$ T) $\wedge$
$\forall h\ t.$
rv_gt0_ninfty $(h::t) \iff$
$(\forall s.\ 0 \le h\ s \wedge h\ s \ne$ PosInf) $\wedge$ rv_gt0_ninfty t

[shared_spare_def]

$\vdash \forall X\ Y\ Z\_a\ Z\_d.$
shared_spare $X\ Y\ Z\_a\ Z\_d$ =
D_OR
(D_OR (D_AND X (D_BEFORE $Z\_d\ X$))
(D_AND X (D_BEFORE $Y\ X$)))
(D_AND Z_a (D_BEFORE $X\ Z\_a$))

[UNIONL_def]

$\vdash$ (UNIONL [] = $\{\}$) $\wedge \forall s\ ss.$ UNIONL $(s::ss)$ = $s \cup$ UNIONL ss

[WSP_def]

$\vdash \forall Y\ X\_a\ X\_d.$
WSP $Y\ X\_a\ X\_d$ =
D_OR
(D_OR
(D_OR (D_AND X_a (D_BEFORE $Y\ X\_a$))
(D_AND Y (D_BEFORE $X\_d\ Y$))) (D_SIMULT $Y\ X\_a$))
(D_SIMULT $Y\ X\_d$)

1.2 Theorems

[AND_BEFORE_ABSORB]

$\vdash \forall X\ Y\ Z.$
D_AND (D_AND (D_BEFORE $X\ Y$) (D_BEFORE $Y\ Z$))
(D_BEFORE $X\ Z$) =
D_AND (D_BEFORE $X\ Y$) (D_BEFORE $Y\ Z$)

[AND_INTER]

$\vdash \forall p\ t\ X\ Y.$
DFT_event p (D_AND $X\ Y$) t =
DFT_event $p\ X\ t \cap$ DFT_event $p\ Y\ t$

[BEFORE_event_GT0]

```
⊢ ∀X Y p t.
      (∀s. 0 ≤ X s) ⇒
      (DFT_event p (D_BEFORE X Y) t =
       {s | X s ≤ Normal t ∧ 0 ≤ X s ∧ X s < Y s} ∩ p_space p)
```

[BEFORE_PREIMAGE_event_GT0]

```
⊢ ∀X Y p t.
      rv_gt0_ninfty [X; Y] ∧ 0 ≤ t ⇒
      (DFT_event p (D_BEFORE X Y) t =
       PREIMAGE (λs. (real (X s),real (Y s)))
         {(w,u) | 0 ≤ w ∧ w < u ∧ w ≤ t} ∩ p_space p)
```

[CDF_prob_DFT_event]

```
⊢ ∀X p t.
      (∀s. X s ≠ PosInf ∧ 0 ≤ X s) ⇒
      (CDF p (λs. real (X s)) t = prob p (DFT_event p X t))
```

[D_ABSORB_AND]

```
⊢ ∀X Y. D_AND X (D_OR X Y) = X
```

[D_AND_ALWAYS]

```
⊢ ∀X. (∀s. 0 ≤ X s) ⇒ (D_AND X ALWAYS = X)
```

[D_AND_ASSOC]

```
⊢ ∀X Y Z. D_AND (D_AND X Y) Z = D_AND X (D_AND Y Z)
```

[D_AND_COMM]

```
⊢ ∀X Y. D_AND X Y = D_AND Y X
```

[D_AND_ID]

```
⊢ ∀X. D_AND X X = X
```

[D_AND_NEVER]

```
⊢ ∀X. D_AND X NEVER = NEVER
```

[D_AND_OVER_OR]

```
⊢ ∀X Y Z. D_AND X (D_OR Y Z) = D_OR (D_AND X Y) (D_AND X Z)
```

[D_AND_SIMULT_ABSORB]

```
⊢ ∀X Y Z.
      D_AND (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
        (D_SIMULT X Z) =
      D_AND (D_SIMULT X Y) (D_SIMULT Y Z)
```

[D_BEFORE_NON_ASSOC1]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_BEFORE Y Z) =
      D_OR (D_BEFORE X Y)
        (D_AND (D_AND X Y) (D_OR (D_BEFORE Z Y) (D_SIMULT Z Y)))
```

[D_BEFORE_NON_ASSOC2]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_BEFORE Y Z) =
      D_OR (D_BEFORE X Y)
        (D_AND (D_AND X Y) (D_INCLUSIVE_BEFORE Z Y))
```

[D_BEFORE_NON_ASSOC3]

```
⊢ ∀X Y Z.
      D_BEFORE (D_BEFORE X Y) Z =
      D_AND (D_BEFORE X Y) (D_BEFORE X Z)
```

[D_BEFORE_NON_COMM]

```
⊢ ∀X Y. D_AND (D_BEFORE X Y) (D_BEFORE Y X) = NEVER
```

[D_BEFORE_OVER_INCLUSIVE_BEFORE]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_INCLUSIVE_BEFORE Y Z) =
      D_OR (D_BEFORE X Y) (D_AND (D_AND X Y) (D_BEFORE Z Y))
```

[D_BEFORE_OVER_SIMULT]

```
⊢ ∀X Y Z.
      D_BEFORE X (D_SIMULT Y Z) =
      D_OR
        (D_OR
           (D_OR (D_AND X (D_BEFORE Y Z))
              (D_AND X (D_BEFORE Z Y))) (D_BEFORE X Y))
        (D_BEFORE X Z)
```

[D_INCLUSIVE_AND]

```
⊢ ∀X Y.
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE Y X) =
      D_SIMULT X Y
```

[D_INCLUSIVE_BEFORE_def2]

```
⊢ ∀X Y.
      D_INCLUSIVE_BEFORE X Y =
      D_OR (D_BEFORE X Y) (D_SIMULT X Y)
```

[D_INCLUSIVE_ID]

```
⊢ ∀X. D_INCLUSIVE_BEFORE X X = X
```

[D_INCLUSIVE_OR_ABSORB]

```
⊢ ∀X Y.
    D_OR (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE Y X) =
    D_OR X Y
```

[D_LEFT_AND_OVER_INCLUSIVE]

```
⊢ ∀X Y.
    D_AND X (D_INCLUSIVE_BEFORE X Y) = D_INCLUSIVE_BEFORE X Y
```

[D_LEFT_AND_OVER_SIMULT]

```
⊢ ∀X Y. D_AND X (D_SIMULT X Y) = D_SIMULT X Y
```

[D_LEFT_BEFORE_OVER_AND]

```
⊢ ∀X Y Z.
    D_BEFORE X (D_AND Y Z) =
    D_OR (D_BEFORE X Y) (D_BEFORE X Z)
```

[D_LEFT_BEFORE_OVER_OR]

```
⊢ ∀X Y Z.
    D_BEFORE X (D_OR Y Z) =
    D_AND (D_BEFORE X Y) (D_BEFORE X Z)
```

[D_LEFT_INCLUSIVE_OVER_AND]

```
⊢ ∀X Y Z.
    D_INCLUSIVE_BEFORE X (D_AND Y Z) =
    D_OR (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_LEFT_INCLUSIVE_OVER_BEFORE]

```
⊢ ∀X Y Z.
    D_INCLUSIVE_BEFORE X (D_BEFORE Y Z) =
    D_OR
      (D_OR (D_BEFORE X Y)
         (D_AND (D_AND X Y) (D_INCLUSIVE_BEFORE Z Y)))
      (D_AND (D_SIMULT X Y) (D_BEFORE Y Z))
```

[D_LEFT_INCLUSIVE_OVER_INCLUSIVE1]

```
⊢ ∀X Y Z.
    D_INCLUSIVE_BEFORE X (D_INCLUSIVE_BEFORE Y Z) =
    D_OR
      (D_OR (D_BEFORE X Y) (D_AND (D_AND X Y) (D_BEFORE Z Y)))
      (D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z))
```

[D_LEFT_INCLUSIVE_OVER_OR]

```
⊢ ∀X Y Z.
    D_INCLUSIVE_BEFORE X (D_OR Y Z) =
    D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_LEFT_INCLUSIVE_OVER_SIMULT]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE X (D_SIMULT Y Z) =
      D_OR
        (D_OR
           (D_OR
              (D_OR (D_AND X (D_BEFORE Y Z))
                 (D_AND X (D_BEFORE Z Y))) (D_BEFORE X Y))
           (D_BEFORE X Z))
        (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
```

[D_LEFT_OR_INCLUSIVE_ABSORB]

```
⊢ ∀X Y. D_OR X (D_INCLUSIVE_BEFORE X Y) = X
```

[D_LEFT_OR_OVER_INCLUSIVE]

```
⊢ ∀X Y. D_OR Y (D_INCLUSIVE_BEFORE X Y) = D_OR X Y
```

[D_LEFT_OR_OVER_SIMULT]

```
⊢ ∀X Y. D_OR X (D_SIMULT X Y) = X
```

[D_LEFT_SIMULT_OVER_AND1]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_AND Y Z) =
      D_OR
        (D_OR (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
           (D_AND (D_SIMULT X Y) (D_BEFORE Z Y)))
        (D_AND (D_SIMULT X Z) (D_BEFORE Y Z))
```

[D_LEFT_SIMULT_OVER_AND2]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_AND Y Z) =
      D_OR (D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Z Y))
        (D_AND (D_SIMULT X Z) (D_INCLUSIVE_BEFORE Y Z))
```

[D_LEFT_SIMULT_OVER_BEFORE]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_BEFORE Y Z) =
      D_AND (D_SIMULT X Y) (D_BEFORE Y Z)
```

[D_LEFT_SIMULT_OVER_INCLUSIVE_BEFORE]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_INCLUSIVE_BEFORE Y Z) =
      D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z)
```

[D_LEFT_SImULT_OVER_OR]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_OR Y Z) =
      D_OR
        (D_OR (D_AND (D_SIMULT X Y) (D_SIMULT Y Z))
           (D_AND (D_SIMULT X Y) (D_BEFORE Y Z)))
        (D_AND (D_SIMULT X Z) (D_BEFORE Z Y))
```

[D_LEFT_SIMULT_OVER_OR2]

```
⊢ ∀X Y Z.
      D_SIMULT X (D_OR Y Z) =
      D_OR (D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z))
        (D_AND (D_SIMULT X Z) (D_INCLUSIVE_BEFORE Z Y))
```

[D_OR_ALWAYS]

```
⊢ ∀X. (∀s. 0 ≤ X s) ⇒ (D_OR X ALWAYS = ALWAYS)
```

[D_OR_ASSOC]

```
⊢ ∀X Y Z. D_OR (D_OR X Y) Z = D_OR X (D_OR Y Z)
```

[D_OR_COMM]

```
⊢ ∀X Y. D_OR X Y = D_OR Y X
```

[D_OR_ID]

```
⊢ ∀X. D_OR X X = X
```

[D_OR_NEVER]

```
⊢ ∀X. D_OR X NEVER = X
```

[D_OR_OVER_AND]

```
⊢ ∀X Y Z. D_OR X (D_AND Y Z) = D_AND (D_OR X Y) (D_OR X Z)
```

[D_RIGHT_BEFORE_OVER_INCLUSIVE_BEFORE]

```
⊢ ∀X Y Z.
      D_BEFORE (D_INCLUSIVE_BEFORE X Y) Z =
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_BEFORE]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_BEFORE X Y) Z =
      D_AND (D_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_INCLUSIVE]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_INCLUSIVE_BEFORE X Y) Z =
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_SIMULT1]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_SIMULT X Y) Z =
      D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE X Z)
```

[D_RIGHT_INCLUSIVE_OVER_SIMULT2]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_SIMULT X Y) Z =
      D_AND (D_SIMULT X Y) (D_INCLUSIVE_BEFORE Y Z)
```

[D_RIGHT_INCLUSIVE_OVER_SIMULT3]

```
⊢ ∀X Y Z.
      D_INCLUSIVE_BEFORE (D_SIMULT X Y) Z =
      D_SIMULT (D_INCLUSIVE_BEFORE X Z)
        (D_INCLUSIVE_BEFORE Y Z)
```

[D_SIMULT_COMM]

```
⊢ ∀X Y. D_SIMULT X Y = D_SIMULT Y X
```

[D_SIMULT_ID]

```
⊢ ∀X. D_SIMULT X X = X
```

[D_SIMULT_NEVER]

```
⊢ ∀X. D_SIMULT X NEVER = NEVER
```

[DFT_BEFORE_event_GT0]

```
⊢ ∀X Y p t.
      (∀s. 0 ≤ X s) ⇒
      (DFT_event p (D_BEFORE X Y) t =
       PREIMAGE (λs. (X s,Y s))
         {(w,u) | 0 ≤ w ∧ w ≤ Normal t ∧ w < u} ∩ p_space p)
```

[DFT_event_AFTER_PREIMAGE_GT0]

```
⊢ ∀p t X Y.
      rv_gt0_ninfty [X; Y] ∧ 0 ≤ t ⇒
      (DFT_event p (D_AND Y (D_BEFORE X Y)) t =
       PREIMAGE (λx. (real (X x),real (Y x)))
         {(u,w) | u < w ∧ 0 ≤ w ∧ w ≤ t} ∩ p_space p)
```

[DFT_event_AND_BEFORE]

```
⊢ ∀p t X Y.
      DFT_event p (D_AND Y (D_BEFORE X Y)) t =
      {u | X u < Y u ∧ Y u ≤ Normal t} ∩ p_space p
```

[DFT_event_AND_BEFORE_PREIMAGE]

$\vdash \forall p\ t\ X\ Y.$
$\quad$ DFT_event p (D_AND Y (D_BEFORE X Y)) t =
$\quad$ PREIMAGE $(\lambda x.\ (X\ x, Y\ x))$ $\{(u,w) \mid u < w \land w \leq \texttt{Normal}\ t\} \cap$
$\quad$ p_space p

[DFT_event_AND_BEFORE_PREIMAGE_GT0]

$\vdash \forall p\ t\ X\ Y.$
$\quad$ $(\forall s.\ 0 \leq Y\ s) \Rightarrow$
$\quad$ (DFT_event p (D_AND Y (D_BEFORE X Y)) t =
$\quad$ PREIMAGE $(\lambda x.\ (X\ x, Y\ x))$
$\quad\quad$ $\{(u,w) \mid u < w \land 0 \leq w \land w \leq \texttt{Normal}\ t\} \cap$ p_space p)

[DFT_PREIMAGE]

$\vdash \forall X\ p\ t.$
$\quad$ DFT_event p X t =
$\quad$ PREIMAGE X $\{u \mid u \leq \texttt{Normal}\ t\} \cap$ p_space p

[FDEP_OR]

$\vdash \forall X\ Y.$ FDEP X Y = D_OR X Y

[HSP_AND]

$\vdash \forall X\ Y.$ HSP Y X = D_AND Y X

[IN_REST]

$\vdash \forall x\ s.\ x \in \texttt{REST}\ s \iff x \in s \land x \neq \texttt{CHOICE}\ s$

[IN_UNIONL]

$\vdash \forall l\ v.\ v \in \texttt{UNIONL}\ l \iff \exists s.\ \texttt{MEM}\ s\ l \land v \in s$

[INCLUSIVE_BEFORE_NEVER]

$\vdash \forall X.$ D_INCLUSIVE_BEFORE X NEVER = X

[lborel_le]

$\vdash \forall x.\ \{w \mid w \leq x\} \in$ measurable_sets lborel

[LEFT_AND_OVER_BEFORE_ABSORB]

$\vdash \forall X\ Y.$ D_AND X (D_BEFORE X Y) = D_BEFORE X Y

[LEFT_OR_OVER_BEFORE_ABSORB]

$\vdash \forall X\ Y.$ D_OR X (D_BEFORE X Y) = X

[Lemma_64]

$\vdash \forall X\ Y.$
$\quad$ D_OR (D_AND X (D_INCLUSIVE_BEFORE Y X))
$\quad\quad$ (D_AND Y (D_INCLUSIVE_BEFORE X Y)) =
$\quad$ D_AND X Y

[Lemma_65]

```
⊢ ∀X Y.
      D_OR (D_INCLUSIVE_BEFORE X Y)
        (D_AND X (D_INCLUSIVE_BEFORE Y X)) =
      X
```

[Lemma_66]

```
⊢ ∀X Y Z.
      D_AND
        (D_AND (D_INCLUSIVE_BEFORE X Y)
           (D_INCLUSIVE_BEFORE Y Z)) (D_INCLUSIVE_BEFORE X Z) =
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_INCLUSIVE_BEFORE Y Z)
```

[Lemma_67]

```
⊢ ∀X Y.
      D_OR (D_INCLUSIVE_BEFORE X Y) (D_BEFORE X Y) =
      D_INCLUSIVE_BEFORE X Y
```

[Lemma_68]

```
⊢ ∀X Y.
      D_OR (D_INCLUSIVE_BEFORE X Y) (D_SIMULT X Y) =
      D_INCLUSIVE_BEFORE X Y
```

[Lemma_69]

```
⊢ ∀X Y. D_AND (D_BEFORE X Y) (D_SIMULT X Y) = NEVER
```

[Lemma_70]

```
⊢ ∀X Y Z.
      D_AND (D_BEFORE X Y) (D_SIMULT Y Z) =
      D_AND (D_BEFORE X Z) (D_SIMULT Y Z)
```

[Lemma_71]

```
⊢ ∀X Y.
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_BEFORE X Y) =
      D_BEFORE X Y
```

[Lemma_72]

```
⊢ ∀X Y. D_AND (D_BEFORE X Y) (D_INCLUSIVE_BEFORE Y X) = NEVER
```

[Lemma_73]

```
⊢ ∀X Y.
      D_AND (D_INCLUSIVE_BEFORE X Y) (D_SIMULT X Y) =
      D_SIMULT X Y
```

[Lemma_74]

```
⊢ ∀X Y.
      D_OR (D_OR (D_BEFORE X Y) (D_SIMULT X Y)) (D_BEFORE Y X) =
      D_OR X Y
```

[Lemma_75]

```
⊢ ∀X Y.
      D_OR (D_OR (D_AND X (D_BEFORE Y X)) (D_SIMULT X Y))
        (D_AND Y (D_BEFORE X Y)) =
      D_AND X Y
```

[Lemma_76]

```
⊢ ∀X Y.
      D_OR (D_OR (D_BEFORE X Y) (D_SIMULT X Y))
        (D_AND X (D_BEFORE Y X)) =
      X
```

[Lemma_77]

```
⊢ ∀X Y Z.
      D_AND (D_AND (D_BEFORE X Y) (D_BEFORE Y Z))
        (D_INCLUSIVE_BEFORE X Z) =
      D_AND (D_BEFORE X Y) (D_BEFORE Y Z)
```

[NEVER_INCLUSIVE_BEFORE_BEFORE]

```
⊢ ∀X. D_INCLUSIVE_BEFORE NEVER X = NEVER
```

[normal_real_mul]

```
⊢ ∀x y.
      x ≠ PosInf ∧ x ≠ NegInf ∧ y ≠ PosInf ∧ y ≠ NegInf ⇒
      (Normal (real x × real y) =
       Normal (real x) × Normal (real y))
```

[OR_UNION]

```
⊢ ∀p t X Y.
      DFT_event p (D_OR X Y) t =
      DFT_event p X t ∪ DFT_event p Y t
```

[PAND_DEF2]

```
⊢ ∀X Y. P_AND X Y = D_AND Y (D_INCLUSIVE_BEFORE X Y)
```

[PIE_lem2_or]

```
⊢ ∀f X Y.
      ALL_DISTINCT [X; Y] ∧
      (∀x. x ∈ {{X}; {Y}; {X; Y}} ⇒ f x ≠ PosInf) ⇒
      (SIGMA f {{X}; {Y}; {X; Y}} = f {X} + f {Y} + f {X; Y})
```

[PREIMAGE_EXTREAL_REAL]

```
⊢ ∀X t p.
      (∀s. X s ≠ PosInf ∧ 0 ≤ X s) ⇒
      (PREIMAGE X {u | u ≤ Normal t} ∩ p_space p =
       PREIMAGE (λs. real (X s)) {u | u ≤ t} ∩ p_space p)
```

[PREIMAGE_EXTREAL_REAL_2RV]

$\vdash \forall X\ Y\ t\ p.$
$\quad (\forall s.\ X\ s \neq \texttt{PosInf} \wedge 0 \leq X\ s \wedge Y\ s \neq \texttt{PosInf} \wedge 0 \leq Y\ s) \wedge$
$\quad 0 \leq t \Rightarrow$
$\quad (\texttt{PREIMAGE}\ (\lambda s.\ (X\ s, Y\ s))$
$\quad\quad \{(u,w) \mid u < w \wedge 0 \leq w \wedge w \leq \texttt{Normal}\ t\} \cap \texttt{p\_space}\ p =$
$\quad \texttt{PREIMAGE}\ (\lambda s.\ (\texttt{real}\ (X\ s), \texttt{real}\ (Y\ s)))$
$\quad\quad \{(u,w) \mid u < w \wedge 0 \leq w \wedge w \leq t\} \cap \texttt{p\_space}\ p)$

[PREIMAGE_EXTREAL_REAL_2RV_BEFORE]

$\vdash \forall X\ Y\ t\ p.$
$\quad (\forall s.\ X\ s \neq \texttt{PosInf} \wedge 0 \leq X\ s \wedge Y\ s \neq \texttt{PosInf} \wedge 0 \leq Y\ s) \wedge$
$\quad 0 \leq t \Rightarrow$
$\quad (\texttt{PREIMAGE}\ (\lambda s.\ (X\ s, Y\ s))$
$\quad\quad \{(w,u) \mid 0 \leq w \wedge w < u \wedge w \leq \texttt{Normal}\ t\} \cap \texttt{p\_space}\ p =$
$\quad \texttt{PREIMAGE}\ (\lambda s.\ (\texttt{real}\ (X\ s), \texttt{real}\ (Y\ s)))$
$\quad\quad \{(w,u) \mid 0 \leq w \wedge w < u \wedge w \leq t\} \cap \texttt{p\_space}\ p)$

[PROB_AND]

$\vdash \forall p\ t\ X\ Y.$
$\quad \texttt{rv\_gt0\_ninfty}\ [X;\ Y] \wedge$
$\quad \texttt{indep\_var}\ p\ \texttt{lborel}\ (\lambda s.\ \texttt{real}\ (X\ s))\ \texttt{lborel}$
$\quad\quad (\lambda s.\ \texttt{real}\ (Y\ s)) \Rightarrow$
$\quad (\texttt{prob}\ p\ (\texttt{DFT\_event}\ p\ (\texttt{D\_AND}\ X\ Y)\ t) =$
$\quad \texttt{CDF}\ p\ (\lambda s.\ \texttt{real}\ (X\ s))\ t \times \texttt{CDF}\ p\ (\lambda s.\ \texttt{real}\ (Y\ s))\ t)$

[PROB_DFT_AFTER]

$\vdash \forall X\ Y\ p\ fy\ t.$
$\quad \texttt{rv\_gt0\_ninfty}\ [X;\ Y] \wedge 0 \leq t \wedge \texttt{prob\_space}\ p \wedge$
$\quad \texttt{indep\_var}\ p\ \texttt{lborel}\ (\lambda s.\ \texttt{real}\ (X\ s))\ \texttt{lborel}$
$\quad\quad (\lambda s.\ \texttt{real}\ (Y\ s)) \wedge$
$\quad \texttt{distributed}\ p\ \texttt{lborel}\ (\lambda s.\ \texttt{real}\ (Y\ s))\ fy \wedge$
$\quad (\forall y.\ 0 \leq fy\ y) \wedge \texttt{cont\_CDF}\ p\ (\lambda s.\ \texttt{real}\ (X\ s)) \wedge$
$\quad \texttt{measurable\_CDF}\ p\ (\lambda s.\ \texttt{real}\ (X\ s)) \Rightarrow$
$\quad (\texttt{prob}\ p\ (\texttt{DFT\_event}\ p\ (\texttt{D\_AND}\ Y\ (\texttt{D\_BEFORE}\ X\ Y))\ t) =$
$\quad \texttt{pos\_fn\_integral}\ \texttt{lborel}$
$\quad\quad (\lambda y.$
$\quad\quad\quad fy\ y \times$
$\quad\quad\quad (\texttt{indicator\_fn}\ \{u \mid 0 \leq u \wedge u \leq t\}\ y \times$
$\quad\quad\quad \texttt{CDF}\ p\ (\lambda s.\ \texttt{real}\ (X\ s))\ y)))$

[PROB_DFT_BEFORE]

$\vdash \forall X\ Y\ p\ fx\ t.$
$\quad \texttt{rv\_gt0\_ninfty}\ [X;\ Y] \wedge 0 \leq t \wedge \texttt{prob\_space}\ p \wedge$
$\quad \texttt{indep\_var}\ p\ \texttt{lborel}\ (\lambda s.\ \texttt{real}\ (X\ s))\ \texttt{lborel}$
$\quad\quad (\lambda s.\ \texttt{real}\ (Y\ s)) \wedge$
$\quad \texttt{distributed}\ p\ \texttt{lborel}\ (\lambda s.\ \texttt{real}\ (X\ s))\ fx \wedge$
$\quad (\forall x.\ 0 \leq fx\ x) \wedge \texttt{measurable\_CDF}\ p\ (\lambda s.\ \texttt{real}\ (Y\ s)) \Rightarrow$

```
        (prob p (DFT_event p (D_BEFORE X Y) t) =
         pos_fn_integral lborel
           (λx.
                fx x ×
                (indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
                 (1 - CDF p (λs. real (Y s)) x))))
```

[PROB_OR]

```
 ⊢ ∀X Y p t.
        rv_gt0_ninfty [X; Y] ∧ All_distinct_events p [X; Y] t ∧
        indep_var p lborel (λs. real (X s)) lborel
          (λs. real (Y s)) ⇒
        (prob p (DFT_event p (D_OR X Y) t) =
         CDF p (λs. real (X s)) t + CDF p (λs. real (Y s)) t -
         CDF p (λs. real (X s)) t × CDF p (λs. real (Y s)) t)
```

[RIGHT_BEFORE_OVER_AND]

```
 ⊢ ∀X Y Z.
        D_BEFORE (D_AND X Y) Z =
        D_AND (D_BEFORE X Z) (D_BEFORE Y Z)
```

[RIGHT_BEFORE_OVER_OR]

```
 ⊢ ∀X Y Z.
        D_BEFORE (D_OR X Y) Z =
        D_OR (D_BEFORE X Z) (D_BEFORE Y Z)
```

[RIGHT_BEFORE_OVER_SIMULT1]

```
 ⊢ ∀X Y Z.
        D_BEFORE (D_SIMULT X Y) Z =
        D_AND (D_SIMULT X Y) (D_BEFORE X Z)
```

[RIGHT_BEFORE_OVER_SIMULT2]

```
 ⊢ ∀X Y Z.
        D_BEFORE (D_SIMULT X Y) Z =
        D_AND (D_SIMULT X Y) (D_BEFORE Y Z)
```

[RIGHT_BEFORE_OVER_SIMULT3]

```
 ⊢ ∀X Y Z.
        D_BEFORE (D_SIMULT X Y) Z =
        D_SIMULT (D_BEFORE X Z) (D_BEFORE Y Z)
```

[RIGHT_OR_OVER_BEFORE_ABSORB]

```
 ⊢ ∀X Y. D_OR (D_BEFORE X Y) Y = D_OR X Y
```

[WSP_CSP]

```
 ⊢ ∀X_a X_d Y.
        (∀s. ALL_DISTINCT [X_a s; Y s]) ∧ (X_d = NEVER) ⇒
        (WSP Y X_a X_d = CSP Y X_a)
```

[WSP_HSP]

```
 ⊢ ∀X Y. WSP Y X X = HSP Y X
```